UNIT-IV (Multivariable Calculus-I)

Introduction :→ In this unit we will study about double & triple integrals, which are very useful in finding area, volume, mass, centroid etc.

Double Integral :→ An integral of the form

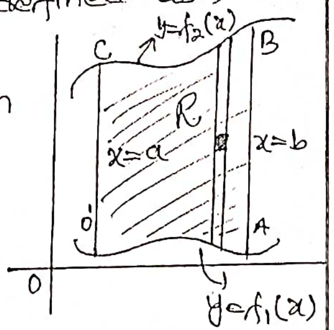
$J = \iint_R f(x,y)\,dx\,dy$ is called double integral of $f(x,y)$ over the region R, which can also be written as

$J = \iint_R f(x,y)\,dy\,dx$.

Evaluation of Double Integral ⇒ The method of evaluating the double integrals depend upon the nature of the curves bounding the region R.

case-a) ⇒ If the region R is defined as ⇒ $a \le x \le b$, $f_1(x) \le y \le f_2(x)$ the region

Here region R is represented by O'ABC, first we will take a vertical strip & treating x as a constant. Then we have lower limit of x is a



lower limit of y is $f_1(x)$
upper limit of x is b
& upper limit of y is $f_2(x)$.

Now the double integral is evaluated first w.r.t y (treating x as a constant) & then w.r.t x, then we have

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=f_1(x)}^{f_2(x)} f(x,y)\,dy\right]dx$$

Ques → Evaluate $\int_0^a \int_0^x xy\,dy\,dx$ [2013-14]

Sol^n ⇒ here $[0=f_1(x)] \le y\ [x=f_2(x)]$
& $[0 \le x \le a]$

so $J = \int_{x=0}^{a}\int_{y=0}^{x} xy\,dy\,dx$

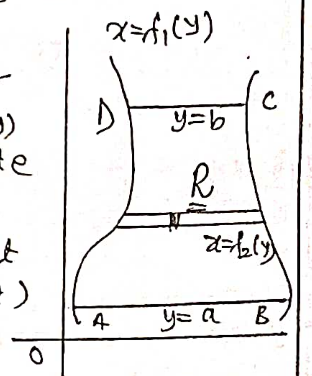
$= \int_{x=0}^{a} x\left[\int_{y=0}^{x} y\,dy\right]dx = \int_{x=0}^{a} x\left(\frac{y^2}{2}\right)_0^x dx$

$= \int_{x=0}^{a} x\cdot\left[\frac{x^2}{2}-0\right]dx = \frac{1}{2}\int_{x=0}^{a} x^3\,dx$

$= \frac{1}{2}\left(\frac{x^4}{4}\right)_0^a = \frac{a^4}{8}$ Ans.

Case: b) ⇒ If Region R is defined as $a \le y \le b$, $f_1(y) \le x \le f_2(y)$

Here region R is bounded by the boundaries $y=a, y=b, x=f_1(y)$ & $x=f_2(y)$, Now to integrate we take a horizontal strip in R & integrate first w.r.t x (by keeping y as a constant) & then integrate w.r.t



y, so the integral becomes

$$\iint_R f(x,y)\,dx\,dy = \int_{y=a}^{y=b}\left[\int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\right]dy$$

Ques: Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

Sol^n: Here $0 \le y \le a$ & $0 \le x \le \sqrt{a^2-y^2}$ so we integrate first w.r.t x.

$$I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\,dx\right]dy$$

$$= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{(a^2-y^2)}{2}\sin^{-1}\frac{x}{\sqrt{a^2-y^2}}\right]_0^{\sqrt{a^2-y^2}} dy$$

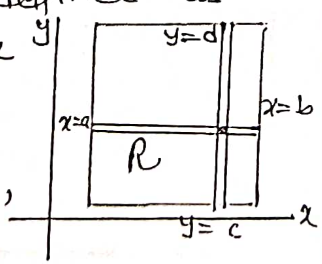
$$\boxed{\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}}$$

$$= \int_0^a \frac{a^2-y^2}{2}\sin^{-1}1\,dy = \frac{\pi}{4}\int_0^a (a^2-y^2)\,dy$$

$$= \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{\pi a^3}{6} \quad \text{Ans.}$$

Case: c $\Rightarrow$ If the Region R is defined as $a \le x \le b$ & $c \le y \le d$ (where a,b,c & d are constants). In this case the order of integration is immaterial, provided that the limits of



integrations are changed accordingly.

Ques: Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

Sol^n: Here $0 \le x \le 1$ & $0 \le y \le 1$ so both variables have constant limits.

$$I = \int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\left[\int_0^1 \frac{1}{\sqrt{1-y^2}}\,dy\right]dx$$

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\left[\sin^{-1}y\right]_0^1 dx = \int_0^1 \frac{1}{\sqrt{1-x^2}}\left[\frac{\pi}{2}-0\right]dx$$

$$= \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}(\sin^{-1}x)_0^1 = \frac{\pi}{2}\times\frac{\pi}{2}$$

$$= \frac{\pi^2}{4} \quad \text{Ans.}$$

Problems on Double Integral

Exp: Evaluate $\int_0^2 \int_0^1 (x^2+3y^2)\,dy\,dx$ [2019-20]

Sol^n: $I = \int_0^2 \left[\int_0^1 (x^2+3y^2)\,dy \right] dx = \int_0^2 \left[x^2y + \frac{3y^3}{3} \right]_0^1 dx$

$$= \int_0^2 [x^2+1]\,dx = \left(\frac{x^3}{3} + x \right)_0^2 = \frac{8}{3}+2$$

$$= \frac{8+6}{3} = \frac{14}{3} \quad \text{Ans.}$$

Exp: Evaluate $\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy$ [2018-19]

Sol^n: Given $I = \int_0^1 \int_0^{x^2} e^{y/x}\,dy\,dx$

here $0 \le y \le x^2$ & $0 \le x \le 1$

$$I = \int_0^1 \left(\frac{e^{y/x}}{1/x} \right)_0^{x^2} dx = \int_0^1 x \left(e^{y/x} \right)_0^{x^2} dx$$

$$= \int_0^1 x\left[e^x - 1\right] dx$$

$$= \int_0^1 [xe^x - x]\,dx = \left[xe^x - \int 1 \times e^x\,dx - \frac{x^2}{2} \right]_0^1$$

$$= \left[xe^x - e^x - \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2}$$

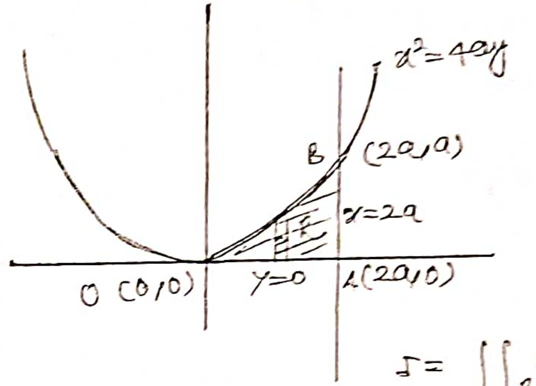
$\boxed{e^0 = 1 \ \& \ e^1 = e}$

Imp

Exp: Find the value of the integral $\iint xy\,dx\,dy$, where R is the region bounded by the x-axis, the line $x=2a$ & the parabola $x^2=4ay$.

Sol: Here region R is the common part bounded by x-axis, $x=2a$ & $x^2=4ay$

[2011-12]



Intersection of $x^2=4ay$ & $x=2a$ is B $(2a,a)$.

R is the region OAB, which is given by

$0 \le x \le 2a$ &

$0 \le y \le \frac{x^2}{4a}$

so Given integral is

$$I = \iint_R xy\,dx\,dy = \int_{x=0}^{2a} \int_{y=0}^{\frac{x^2}{4a}} xy\,dy\,dx$$

$$= \int_{x=0}^{2a} x \left(\frac{y^2}{2} \right)_0^{\frac{x^2}{2a}} dx = \int_0^{2a} \frac{x^5}{32a^2}\,dx$$

$$= \frac{1}{32a^2} \left(\frac{x^6}{6} \right)_0^{2a} = \frac{a^4}{3} \quad \text{An.}$$

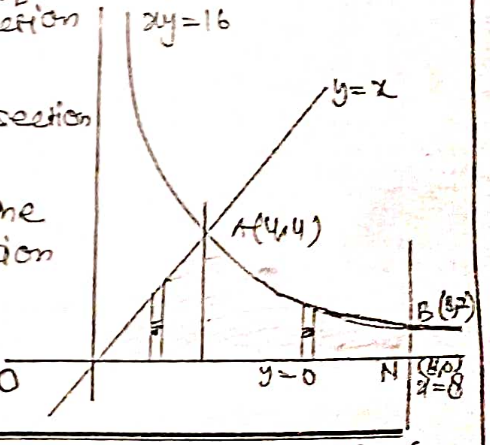
Imp

Exp: Let D be the region in the first Quadrant bounded by the curves $xy=16$, $x=y$, $y=0$ & $x=8$. Sketch the region of integration of the following integral $\iint_D x^2\,dx\,dy$ & evaluate it.

Sol^n: $A(4,4)$ is the intersection of $xy=16$ & $y=x$.

$B(8,2)$ is the intersection of $xy=16$ & $x=8$.

Shaded portion is the region of integration



Given

$I = \iint_D x^2\,dx\,dy$

To evaluate the given integral, we divide the area OABNO into two parts by AM as shown in the figure.

so $I = \iint_D x^2 dx\, dy = \iint_{OMAO} x^2 dx\, dy + \iint_{MNBAM} x^2 dx\, dy$

$$= \int_{x=0}^{4} \int_{y=0}^{x} x^2 dy\, dx + \int_{x=4}^{8} \int_{y=0}^{\frac{16}{x}} x^2 dy\, dx$$

$$= \int_0^4 x^2 (y)_0^x dx + \int_4^8 x^2 (y)_0^{16/x} dx$$

$$= \int_0^4 x^3 dx + \int_4^8 16x\, dx = \left(\frac{x^4}{4}\right)_0^4 + (8x^2)_4^8$$

$= 64 + 8(64-16)$

$= 64 + 384$

$= 448$ Ans

For Practice

Ques: Evaluate $\int_0^1 \int_0^{x^2} x e^y\, dy\, dx$ [2017-10]

Double integral in Polar Coordinates

Introduction: Double integrals of the form $I = \int_{\theta=\theta_1}^{\theta_2} \int_{r=r_1}^{r_2} f(r,\theta)\, dr\, d\theta$ is known as double integral in Polar coordinates. To solve these types of integrals we first solve w.r.t. r between the limits r_1 & r_2 & then solve w.r.t θ between θ_1 to θ_2.

Exp: Evaluate $\int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a} r^2 dr\, d\theta$.

Sol: we have $I = \int_{\theta=0}^{\pi/2} \int_{r=a(1-\cos\theta)}^{a} r^2 dr\, d\theta$.

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^3}{3}\right]_{a(1-\cos\theta)}^{a} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{a^3}{3} - \frac{a^3(1-\cos\theta)^3}{3}\right] d\theta.$$

$$= \frac{a^3}{3} \int_{\theta=0}^{\pi/2} \left[1-(1-\cos\theta)^3\right] d\theta.$$

$$= \frac{a^3}{3} \int_0^{\pi/2} \left[1-(1-3\cos\theta+3\cos^2\theta-\cos^3\theta)\right] d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/2} \left[3\cos\theta - 3\cos^2\theta + \cos^3\theta\right] d\theta.$$

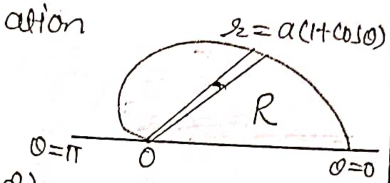
$$= \frac{a^3}{3} \int_0^{\pi/2} \left[3\cos\theta - 3\left(\frac{1+\cos 2\theta}{2}\right) + \frac{1}{4}(\cos 3\theta + 3\cos\theta\right] d\theta.$$

$$= \frac{a^3}{3} \left[3\sin\theta - \frac{3}{2}\theta - \frac{3}{2}\frac{\sin 2\theta}{2} + \frac{1}{4}\frac{\sin 3\theta}{3} + 3\sin\theta\right]_0^{\pi/2} = \frac{a^3}{36}[44-9\pi]$$

V.Imp

Exp: Evaluate $\iint r\sin\theta \, dr\, d\theta$ over the area of the cardioid $r = a(1+\cos\theta)$ above the initial line.

Sol^n: Region 'R' of integration is given by



$0 \le \theta \le \pi$

& $0 \le r \le a(1+\cos\theta)$.

$$I = \iint_R r\sin\theta \, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{0=r}^{a(1+\cos\theta)} r\sin\theta \, dr\, d\theta.$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left(\frac{r^2}{2}\right)_0^{a(1+\cos\theta)} d\theta.$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi} \sin\theta \; a^2(1+\cos\theta)^2 \, d\theta$$

putting $1+\cos\theta = t \Rightarrow \sin\theta\, d\theta = -dt$

$\theta = 0 \Rightarrow t = 2$

$\theta = \pi \Rightarrow t = 0$

$$I = \frac{1}{2}\int_2^0 a^2 t^2 (-dt) = \frac{a^2}{2}\int_0^2 t^2 dt$$

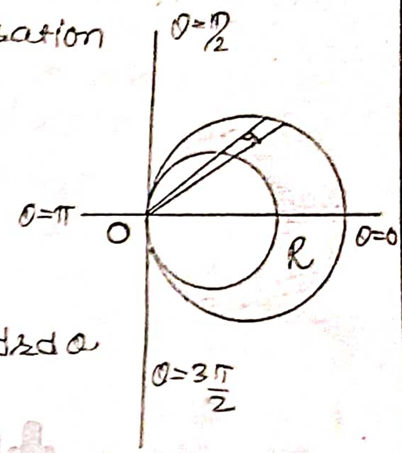
$$= \frac{a^2}{2}\left(\frac{t^3}{3}\right)_0^2 = \frac{a^2}{2} \times \frac{8}{3}$$

$$= \frac{4a^2}{3} \quad \text{Ans}$$

V.Imp

Exp: Evaluate $\iint r^3 dr\, d\theta$, over the area bounded between the circles $r = 2\cos\theta$ & $r = 4\cos\theta$.

Sol^n: Region R of integration is given by



$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ &

$2\cos\theta \le r \le 4\cos\theta$,

$$I = \iint_R r^3 \, dr\, d\theta.$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=2\cos\theta}^{4\cos\theta} r^3 \, dr\, d\theta$$

$$I = \int_{-\pi/2}^{\pi/2} \left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta.$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{4}\left[256\cos^4\theta - 16\cos^4\theta\right] d\theta$$

$$= 120 \int_0^{\pi/2} \cos^4\theta \, d\theta.$$

$$= 120 \times \frac{3\times 1}{4\times 2} \times \frac{\pi}{2} = \frac{45}{2}\pi$$

formula $\int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = \frac{1\cdot 3\cdot 5 \cdots (n-1)}{2\cdot 4\cdot 6 \cdots n} \cdot \frac{\pi}{2}$

If n is even

$\int_0^{\pi/2} \cos^n\theta \, d\theta = \int_0^{\pi/2} \sin\theta \, d\theta = \frac{2\cdot 4\cdot 6 \cdots (n-1)}{1\cdot 3\cdot 5 \cdots n}$, If n is odd

Change of order of Integration

Introduction: On changing the order of integration, the limits of integration change. To find the new limits, we draw the rough sketch of the region of integration. Value of integration is unchanged by changing of the order of integration.

Some complicated integrals can be made easy to handle by a change in the order of integration.

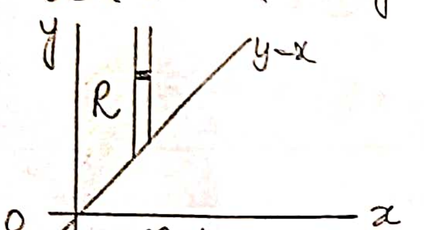
Exp: Evaluate the following integral by changing the order of integration, $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

[2020-21]

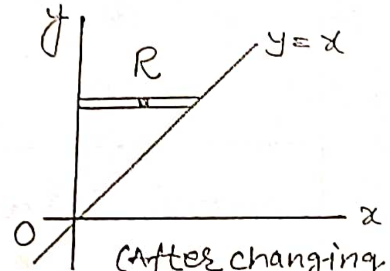
Sol^n: Given integral is $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$

Given region of integration is

$0 \le x < \infty$ & $x \le y \le \infty$



(Before changing the order)



(After changing the order)

After changing of the order of Integration R can be written as $0 \le x \le y,\ 0 \le y < \infty$

So $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ (Before)

$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$ (After)

$= \int_0^\infty \left(\frac{e^{-y}}{y}\right)\int_0^y (1)\,dx\,dy$

$= \int_0^\infty \frac{e^{-y}}{y}\,(x)_0^y\,dy = \int_0^\infty \frac{e^{-y}}{y}\times y\,dy$

$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = \left(-\frac{1}{e^y}\right)_0^\infty = -\left[\frac{1}{\infty} - 1\right] = -(0-1)$

$= 1$ Ans.

V.V. Imp.

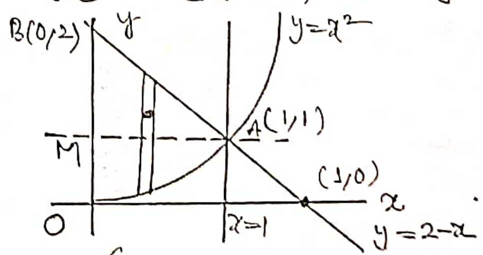
Exp. Evaluate the integration by changing of order of $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$

[2014-15, 2015-16, 2016-17, 2017-18, 2019-20]

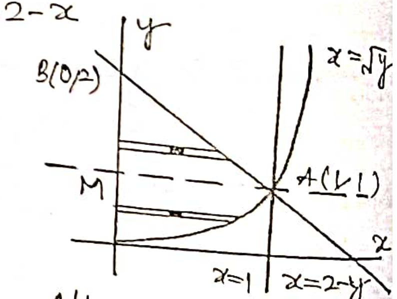
Sol^n: Given $I = \int_0^1 \int_{x^2}^{2-x} xy\,dy\,dx$ — ①

where region of integral is bounded by the curves $y = x^2,\ y = 2-x,\ x = 0$ & $x = 1$

i.e $0 \le x \le 1,\ x^2 \le y \le 2-x$



Given Region



After changing the order

After changing the order we have

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \iint_{OAMO} xy\,dx\,dy + \iint_{MABM} xy\,dx\,dy$$

$$= \int_0^1\int_0^{\sqrt{y}} xy\,dx\,dy + \int_1^2\int_0^{2-y} xy\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 y\,(x^2)_0^{\sqrt{y}}\,dy + \int_1^2 y\left(\frac{x^2}{2}\right)_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2\,dy + \frac{1}{2}\int_1^2 y\,(2-y)^2\,dy$$

$$= \frac{1}{2}\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}\int_1^2 [4y - 4y^2 + y^3]\,dy$$

$$= \frac{1}{6} + \frac{1}{2}\left[2y^3 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

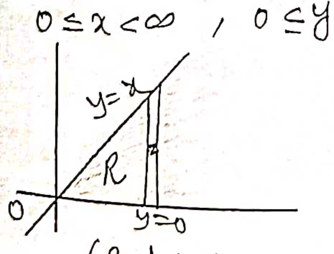
$$= \frac{1}{6} + \frac{1}{2}\left[\left(8 - \frac{32}{3} + 4\right) - \left(2 - \frac{4}{3} + \frac{1}{4}\right)\right]$$

$$= \frac{3}{8} \quad \text{Ans.}$$

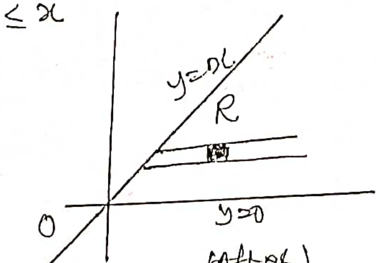
Exp: Evaluate the integral $\int_0^\infty\int_0^x x\cdot\exp\left(-\frac{x^2}{y}\right)dx\,dy$ by changing the order of integration.

Soln. Region of Given integral is bounded by

$0 \le x < \infty$, $0 \le y \le x$



(Before)



(After)

After changing the order, region is given by

$y \le x < \infty$, $0 \le y < \infty$

$$I = \int_0^\infty\int_0^x x\cdot\exp\left(-\frac{x^2}{y}\right)dy\,dx \quad \text{(Before)}$$

$$= \int_0^\infty\int_y^\infty x\,e^{-(x^2/y)}\,dx\,dy \quad \text{(After)}$$

$$= \int_0^\infty \left[\int_y^\infty x e^{-x^2/y}\,dx\right] dy$$

$$= \int_0^\infty \left[-\frac{y}{2}e^{-\frac{x^2}{y}}\right]_y^\infty dy = \int_0^\infty \frac{y}{2}e^{-y}\,dy$$

$$= \left[\frac{y}{2}(-e^{-y}) - \frac{1}{2}(e^{-y})\right]_0^\infty = \left[(0-0)-\left(0-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2} \quad \text{Ans.}$$

Questions for Practice →

Ques 1: Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{(\sqrt{1-x^2-y^2})(e^y+1)}\,dx\,dy$ [2011-12]

Ques 2: Changing the order of integration in $I = \int_0^\infty\int_{x/4}^{2} f(x,y)\,dy\,dx$ leads to [2016-17]

$I = \int_r^s\int_p^q f(x,y)\,dx\,dy$, say, what is q?

Ques 3: Change the order of integration & evaluate $\int_0^2\int_{x^2/4}^{3-x} xy\,dy\,dx$ [2018-19]

Area by Double Integral

Introduction: Area by double integration is given by

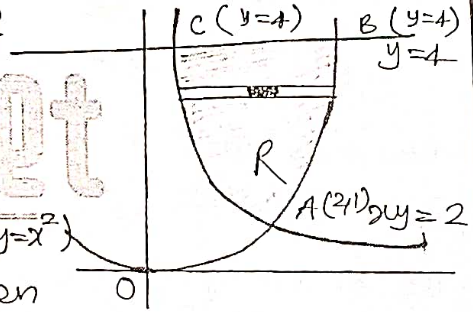
a) Area of region $R = \iint_R dx\,dy$ [in Cartesian]

b) Area of region $R = \iint_R r\,dr\,d\theta$ [in Polar]

Exp: Determine the area bounded by the curves $xy = 2$, $4y = x^2$ & $y = 4$ in I^{st} quadrant. [2014-15]

Sol^n: Required area of shaded region is

$A = \iint_R dx\,dy$ —(1)



where R is given by $\Rightarrow$ $1 \le y \le 4$, $\frac{2}{y} \le x \le 2\sqrt{y}$

$$A = \int_{y=1}^{4} \int_{x=\frac{2}{y}}^{2\sqrt{y}} dx\,dy = \int_1^4 \left[2\sqrt{y} - \frac{2}{y}\right] dy$$

$$= 2\left[\frac{2}{3} y^{3/2} - \log y\right]_1^4$$

$$= 2\left[\frac{16}{3} - 2\log 2 - \frac{2}{3}\right]$$

$$= \frac{28}{3} - 4\log 2 \quad \text{(Ans)}.$$

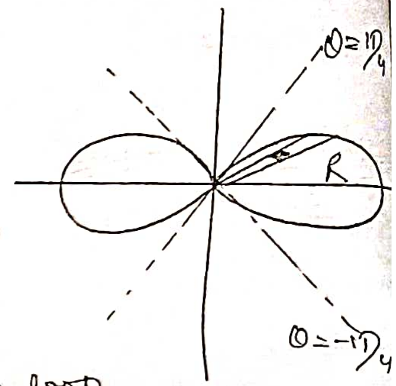
Exp: Compute the area of lemniscate $r^2 = a^2\cos 2\theta$ [2013-14]

Sol^n: Given curve is

$r^2 = a^2 \cos 2\theta$

or $r = \pm a\sqrt{\cos 2\theta}$

here θ lies between $\theta = 0$ to $\theta = \pi/4$.



Required area

$A = 4 \times$ Area of one loop

$$= 4 \iint_R r\,dr\,d\theta$$

$$A = 4 \int_{\theta=0}^{\pi/4} \int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta.$$

$$= 4 \int_0^{\pi/4} \left(\frac{r^2}{2}\right)_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 4 \times \frac{1}{2} \int_0^{\pi/4} \left(a\sqrt{\cos 2\theta}\right)^2 d\theta.$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta\,d\theta$$

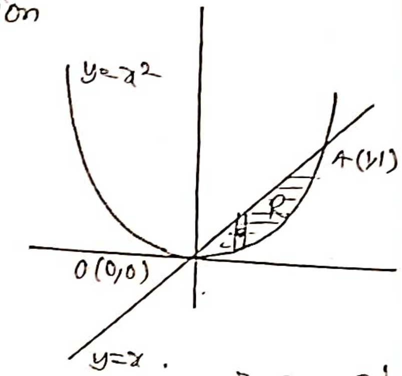
$$= 2a^2 \left(\frac{\sin 2\theta}{2}\right)_0^{\pi/4}$$

$$= \frac{2a^2}{2}\left[\sin \pi/2 - 0\right]$$

$$= a^2 \quad \text{Ans}.$$

Exp: Evaluate the area enclosed between the parabola $y=x^2$ & the straight line $y=x$.

Sol^n: Area of the shaded region



$$A=\iint_R dx\,dy$$

$$=\int_{x=0}^{1}\int_{y=x^2}^{x} dy\,dx$$

$$=\int_0^1 [y]_{x^2}^{x}\,dx$$

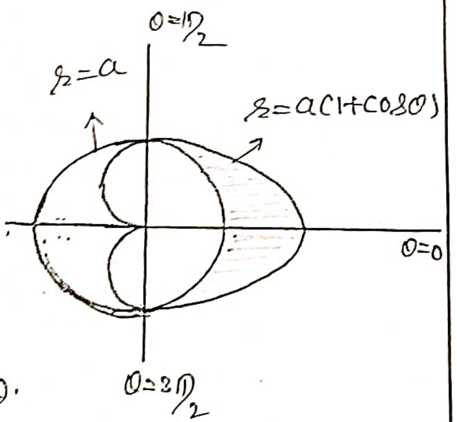
$$=\int_0^1 [x-x^2]\,dx=\int_0^1 [x-x^2]\,dx=\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}\ \text{Ans.}$$

Imp.

Exp: Find the area outside the circle $r=a$ & inside the cardioid $r=a[1+\cos\theta]$

Sol^n: Area of the shaded region is



$$A=\iint_R r\,dr\,d\theta$$

$$=2\int_{\theta=0}^{\pi/2}\int_{r=a}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$=2\int_0^{\pi/2}\frac{1}{2}\left[a^2(1+\cos\theta)^2-a^2\right]d\theta$$

$$=a^2\int_0^{\pi/2}\left[(1+2\cos\theta+\cos^2\theta-1)\right]d\theta$$

$$\text{Area}=a^2\int_0^{\pi/2}\left[2\cos\theta+\left(\frac{1+\cos2\theta}{2}\right)\right]d\theta.$$

$$=\frac{a^2}{2}\int_0^{\pi/2}\left[1+4\cos\theta+\cos2\theta\right]d\theta.$$

$$=\frac{a^2}{2}\left[\theta+4\sin\theta+\frac{\sin2\theta}{2}\right]_0^{\pi/2}$$

$$=\frac{a^2}{2}\left[\frac{\pi}{2}+4\right]=\frac{a^2}{4}(\pi+8)\ \text{Ans.}$$

Ques. for Practice:

Ques: Find the area lying between the parabolas $y=4x-x^2$ & above the line $y=x$. [2019-20].

Introduction to Triple Integration, Volume by Triple Integral

Introduction: Triple integral of a function of three variables $f(x,y,z)$ in a region V is denoted by $I = \iiint_V f(x,y,z)\,dx\,dy\,dz$

Evaluation of Triple integral: If region V is defined by $a \le x \le b$, $c \le y \le d$, $g \le z \le h$ where a, b, c, d, g, h all are constants. then

$$I = \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=g}^{h} f(x,y,z)\,dz\,dy\,dx$$

can be calculated first wrt z between g and h, then wrt y between c & d & then wrt x between a & b.

If V is given by $a \le x \le b$, $\phi_1(x) \le y \le \phi_2(x)$ & $f_1(x,y) \le z \le f_2(x,y)$ then

$$I = \int_{x=a}^{b} \int_{y=\phi_1(x)}^{\phi_2(x)} \int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

If z is the inner most integral then we solve the above integral first wrt z by keeping x, y as constant then wrt y by keeping x as a constant & at last wrt x between a to b.

Exp: Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$, where R: $0 \le x \le 1$, $1 \le y \le 2$; $2 \le z \le 3$ [2015-16, 2017

Sol^n: $I = \int_{x=0}^{1} \int_{y=1}^{2} \int_{z=2}^{3} (x+y+z)\,dz\,dy\,dx$

$$= \int_{x=0}^{1} \int_{y=0}^{2} \left[(x+y)z + \frac{z^2}{2}\right]_2^3 dy\,dx$$

$$= \int_0^1 \int_{y=0}^{2} \left[(x+y) + \frac{5}{2}\right] dy\,dx$$

$$= \int_0^1 \left[\left(x+\frac{5}{2}\right)y + \frac{y^2}{2}\right]_1^2 dx = \int_0^1 \left[\left(x+\frac{5}{2}\right)+\frac{3}{2}\right](dx$$

$$= \left(\frac{x^2}{2} + 4x\right)_0^1 = \frac{9}{2} \quad \text{Ans.}$$

Volume by the Triple integral

Volume of the solid 'R' is given by

$$\text{Volume} = \iiint_R dx\,dy\,dz$$

Exp: Imp. Find the volume of the solid bounde by the surfaces $x=y=z=0$ & $x+y+z=1$ [2010-19, 2020-21]

Sol^n: $\text{Volume} = \iiint_R dx\,dy\,dz$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dz\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (z)_0^{1-x-y}\,dy\,dx$$

$$= \int_0^1 \int_{y=0}^{1-x} (1-x-y)\, dy\, dx$$

$$= \int_0^1 \left[(-x+1)y - \frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 \left[(1-x)^2\right] dx$$

$$= \frac{1}{2}\left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{1}{6} \quad \text{Ans.}$$

Imp
Exp: find the volume of the region bounded by the surfaces $y=x^2$, $x=y^2$ & the planes $z=0$, $z=3$ [2019-20]

Sol: $\text{Volume} = \iiint_R dx\, dy\, dz \quad —(1)$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} \int_{z=0}^{3} dz\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} [z]_0^3\, dy\, dx = 3\int_0^1 (y)_{x^2}^{\sqrt{x}}\, dx$$

$$= 3\int_0^1 [\sqrt{x} - x^2]\, dx$$

$$= 3 \int_0^1 \left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)_0^1$$

$$= 3\left(\frac{2}{3} - \frac{1}{3}\right)$$

$$= 3\left[\frac{1}{3}\right]$$

$$= 1 \text{ cubic unit}$$

Question for Practice ⇒

Ques: Evaluate the triple integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$$

[2016-17]

Ques: Evaluate $\iiint x^2yz\, dx\, dy\, dz$ throughout the volume bounded by the planes $x=0$, $y=0$, $z=0$ & $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

[2016-17]

Ques: find the volume of the solid which is bounded by the surfaces $az = x^2+y^2$ & $z=x$ [2011-12]

Change of variables in Double & Triple integral

Change of variables in Double Integral:

Let the double integral

$I = \iint_R f(x,y)\, dx\, dy$ — ① if it is to be changed in the new variables u & v.

Relation between u, v, x & y is given by

$x = \phi(u,v)$, $y = \psi(u,v)$

Then

$I = \iint_R f(x,y)\, dx\, dy$

$= \iint_{R'} f[\phi(u,v), \psi(u,v)]\, |J|\, du\, dv$ — ②

where $dx\, dy = |J|\, du\, dv$ & $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

Change of variables from (x,y) to Polar coordinates (r,θ) ⇒

Here $x = r\cos\theta$, $y = r\sin\theta$

$\iint_R f(x,y)\, dx\, dy = \iint f[r\cos\theta, r\sin\theta]\, r\, dr\, d\theta$

$J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ so $dx\, dy = r\, dr\, d\theta$.

Exp: Evaluate $\iint (x+y)^2\, dx\, dy$, where R is the region bounded by the parallelogram in the xy-plane with vertices $(1,0), (3,1), (2,$ $(0,1)$, using the transformation

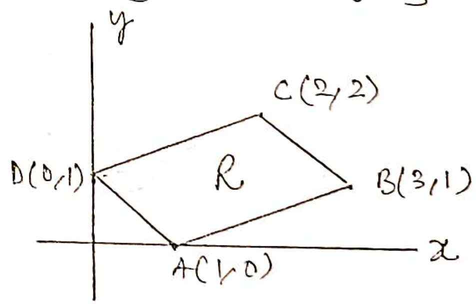
$u = x+y$, $v = x-2y$ [2019-20]

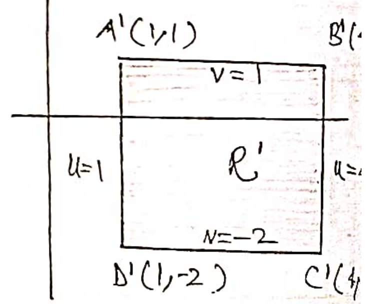
Sol^n: The vertices $A(1,0), B(3,1), C(2,2), D(0,$ of the parallelogram ABCD in xy-plan become $A'(1,1)$, $B'(4,1)$, $C'(4,-2)$ & $D'(1,$ in the uv-plane by using the transformations

$u = x+y$ & $v = x-2y$

The region R in xy plane become the region R' in the uv-plane whi is a rectangle bounded by the lin $u=1$, $u=4$, $v=-2$ & $v=1$. Solving the giv equations for x & y we get

$x = \frac{1}{3}(2u+v)$, $y = \frac{1}{3}(u-v)$,





$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} \end{vmatrix} = -\frac{1}{3}$$

$$dx\,dy = |J|\,du\,dv = \left|-\frac{1}{3}\right| du\,dv = \frac{1}{3}\,du\,dv$$

$$\iint_R (x+y)^2\,dx\,dy = \iint_{R'} u^2\,|J|\,du\,dv$$

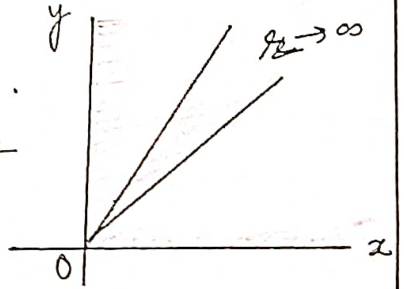
$$= \int_{-2}^{1}\int_{1}^{4} \frac{u^2}{3}\,du\,dv = \int_{-2}^{1} \frac{1}{3}\left(\frac{u^3}{3}\right)_1^4 dv$$

$$= \int_{-2}^{1} 7\,dv = 7 \times 3 = 21 \quad \text{Ans.}$$

Imp

Exp: Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates, hence show that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ [2018-19]

Sol: Given region of integration is Ist quadrant. To change it into polar we have



$x = r\cos\theta,\ y = r\sin\theta.$

$dx\,dy = r\,dr\,d\theta.$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta.$$

$$= \int_0^{\pi/2}\int_0^\infty \frac{e^{-t}}{2}\,dt\,d\theta \quad \text{, where } t = r^2$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}e^{-t}\right]_0^\infty d\theta.$$

$$= -\frac{1}{2}\int_0^{\pi/2} (0-1)\,d\theta.$$

$$= \frac{1}{2}(\theta)_0^{\pi/2}$$

$$= \pi/4$$

Now let $I = \int_0^\infty e^{-x^2}\,dx$

between the same limits, we have

$$I = \int_0^\infty e^{-y^2}\,dy$$

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2}\,dx\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \pi/4$$

$$I = \frac{\sqrt{\pi}}{2}$$

Practice:

Ques: Evaluate by changing the variables $\iint (x+y)^2\,dx\,dy$ where R is the region bounded by the lines $x+y=0$, $x+y=2$, $3x-2y=0$, $3x-2y=3$
[2013-14], [2020-21]

Ques: Evaluate $\iint (x-y)^4 \exp(x+y)\,dx\,dy$, where R is the square in the x-y plane with vertica at (1,0), (2,1), (1,2) & (0,1). [2012-13]

Problems on change of variables in Double & Triple integral:

Exp: Evaluate the following by changing into polar coordinates $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2 \sqrt{x^2+y^2}\,dx\,dy$

Sol^n: changing to polar coordinates

we have $x = r\cos\theta$, $y = r\sin\theta$

so $x^2+y^2=1 \Rightarrow r^2 = 1$

$$I = \int_0^a \int_0^{\sqrt{a^2-y^2}} y^2 \sqrt{x^2+y^2}\,dx\,dy$$

$$= \int_0^{\pi/2} \int_0^a r^2 \sin^2\theta \cdot r \cdot r\,dr\,d\theta.$$

$$= \int_0^{\pi/2} \sin^2\theta \cdot \left(\frac{r^5}{5}\right)_0^a d\theta$$

$$= \frac{a^5}{5} \int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta.$$

$$= \frac{a^5}{10} \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi a^5}{20}$$

Change of variables in Triple integral

$J = \iiint_V f(x,y,z)\, dx\, dy\, dz$, can be changed in to the variables u, v, w as

$$J = \iiint_{V'} f\left[\phi_1(x,y,z), \phi_2(x,y,z), \phi_3(x,y,z)\right] |J|\, du\, dv\, dw$$

where $u = \phi_1(x,y,z)$, $v = \phi_2(x,y,z)$
$w = \phi_3(x,y,z)$

& $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Change of cartesian coordinates (x,y,z) to spherical polar coordinate:

If we have $J = \iiint_R f(x,y,z)\, dx\, dy\, dz$ — (1)

Then to change above integral in spherical polar coordinates, we have

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$

so $dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

so (1) will be

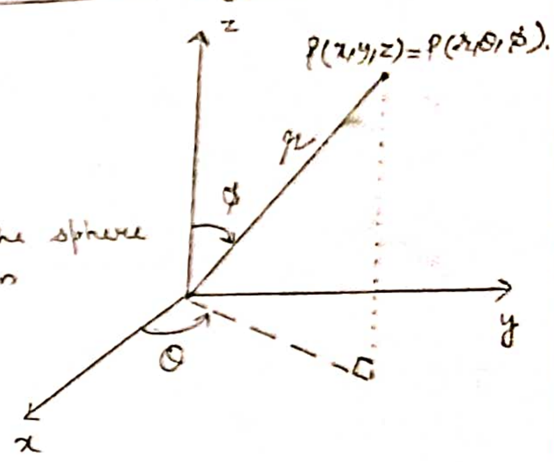
$$J = \iiint_{R'} F(r,\theta,\phi)\, r^2\sin\theta\, dr\, d\theta\, d\phi \quad - (2)$$

Spherical Polar coordinate system:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$



General Equation of the sphere with centre at origin and radius r is:

$x^2+y^2+z^2 = r^2$

Ques: Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar coordinates. [2015-16]

Sol^n: here region of integration is bounded by $0 \le z \le \sqrt{1-x^2-y^2}$,

$0 \le y \le \sqrt{1-x^2}$ & $0 \le x \le 1$

so ($x^2+y^2+z^2 = 1^2$, sphere of radius 1)

region is in I^st quadrant only

$$J = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{r^2\sin\theta}{\sqrt{1-r^2}}\, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \left[\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right] \sin\theta \, dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \left[\sin^{-1} r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1} r\right)\right]_0^1 d\phi\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta \left(\frac{\pi}{2} - \frac{\pi}{4}\right) d\theta\, d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\, d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} d\phi$$

$$= \frac{\pi^2}{8}$$

Question for Practice:

Ques: If the volume of an object expressed in the spherical coordinates as following:

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2 \sin\phi \, dr\, d\phi\, d\theta.$$ Evaluate the value of V. [2016-17]

Lecture-14

Gamma function and its properties

Gamma function :- Gamma function is denoted as $\Gamma(n), n>0$ and defined as

$$\boxed{\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx}$$

Important results :- (1) $\Gamma(n+1) = n\,\Gamma(n)$

(1a) $\Gamma(n+1) = n!$ [if n is +ve inte

(1b) $\Gamma(n) = (n-1)(n-2)\times$ go on decreasing b

(the multiplication of factors being continued so long as the factors remain positive, multiplied by Γ(last factor))

Example:- $\Gamma(5) = 4! = 24$

Example:- Evaluate $\dfrac{\Gamma(8/3)}{\Gamma(2/3)}$. [2015-16]

Solution:-

$$\Gamma(8/3) = \frac{5}{3}\cdot\frac{2}{3}\,\Gamma\left(\frac{2}{3}\right)$$ Using (1b) formula

then $$\frac{\Gamma(8/3)}{\Gamma(2/3)} = \frac{\frac{5}{3}\cdot\frac{2}{3}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{10}{9}$$

(2) $\Gamma(n+1) = n\,\Gamma(n)$ can be used to find the value $\Gamma(\text{negative } n)$ where $n \neq 0, -1, -2, \ldots$

Example:- Find the value of $\Gamma(-\frac{5}{2})$

Sol :- $\Gamma(n+1) = n\,\Gamma(n) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$

$$\Gamma(-\tfrac{5}{2}) = \frac{\Gamma(-\frac{3}{2})}{-\frac{5}{2}} = \frac{\Gamma(-\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})}$$

$$= \frac{\Gamma(\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}$$

$$= -\frac{8}{15}\Gamma(\tfrac{1}{2})$$

(3) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(4) $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma(\frac{m+1}{2})\,\Gamma(\frac{n+1}{2})}{2\,\Gamma(\frac{m+n+2}{2})}$$

(5) $$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Example:- Evaluate $\Gamma(3/4)\,\Gamma(1/4)$.

Sol:- $\Gamma(3/4)\,\Gamma(1/4) = \Gamma(1/4)\,\Gamma(1-1/4) = \frac{\pi}{\sin\frac{\pi}{4}}, \; n = \frac{1}{4}$

$= \sqrt{2}\,\pi$

Example :- Find the value of $\int_0^\infty e^{-ax} x^{n-1} dx$

Sol :- $$I = \int_0^\infty e^{-ax} x^{n-1} dx$$

put $ax = t \Rightarrow a\,dx = dt$ then

$$I = \int_0^\infty e^{-t}\left(\frac{t}{a}\right)^{n-1}\frac{dt}{a} = \frac{1}{a^n}\int_0^\infty e^{-t} t^{n-1}\, dt$$

$$I = \frac{1}{a^n}\Gamma(n) = \int_0^\infty e^{-ax} x^{n-1} dx$$

(6) Duplication formula :- Prove that

$$2^{2n-1}\,\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\,\Gamma(2n)$$

n is positive

Proof:- $\because$ $$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \quad \text{---(1)}$$

putting $2m-1 = 0 \Rightarrow m = \frac{1}{2}$ in (1), we get

$$\int_0^{\pi/2} \sin^{2n-1}\theta\, d\theta = \frac{\Gamma(n)\,\Gamma(\frac{1}{2})}{2\,\Gamma(n+\frac{1}{2})} \quad \text{---(2)}$$

Again putting $m = n$ in (1), we get

$$\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{(\Gamma(n))^2}{2\,\Gamma(2n)}$$

$$\frac{1}{2^{2n-1}}\int_0^{\pi/2} (2\sin\theta\cos\theta)^{2n-1} d\theta = \frac{(\Gamma(n))^2}{2\,\Gamma(2n)}$$

$$\frac{1}{2^{2n}}\int_0^{\pi/2}(\sin 2\theta)^{2n-1}\,2d\theta = \frac{(\Gamma(n))^2}{2\,\Gamma(2n)}$$

Putting $2\theta=\phi$ so that $2d\theta = d\phi$, we get

$$\frac{1}{2^{2n}}\int_0^{\pi}(\sin\phi)^{2n-1}\,d\phi = \frac{(\Gamma(n))^2}{2\,\Gamma(2n)}$$

$\because \sin(\pi-\phi) = \sin\phi$

So

$$\frac{2}{2^{2n}}\int_0^{\pi/2}\sin^{2n-1}\phi\,d\phi = \frac{(\Gamma(n))^2}{2\,\Gamma(2n)}$$

or

$$\int_0^{\pi/2}\sin^{2n-1}\theta\,d\theta = \frac{2^{2n-1}(\Gamma(n))^2}{2\,\Gamma(2n)} \quad —(3)$$

from (2) and (3)

$$\frac{\Gamma(n)\,\Gamma(\frac{1}{2})}{2\,\Gamma(n+\frac{1}{2})} = \frac{2^{2n-1}(\Gamma(n))^2}{2\,\Gamma(2n)}$$

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n)$$

Or

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2})\cdot 2^{2n-1} = \sqrt{\pi}\,\Gamma(2n)$$

Example:- Show that $\dfrac{\Gamma(\frac{1}{3})\,\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})} = 2^{1/3}\sqrt{\pi}$

Solution:- $\because$ we know that

$$\Gamma(n)\,\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n-1}}\,\Gamma(2n) \quad —(1)$$

Putting $n=\frac{1}{3}$ in (1), we get

$$\Gamma(\tfrac{1}{3})\,\Gamma(\tfrac{1}{3}+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{-\frac{1}{3}}}\,\Gamma(\tfrac{2}{3})$$

$$\frac{\Gamma(\frac{1}{3})\,\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3})} = 2^{1/3}\sqrt{\pi}$$ Ans

Practice Questions :-

(1) Show that $\displaystyle\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sqrt{\cot\theta} = \frac{\pi}{\sqrt{2}}$

(ii) Find the value of $\Gamma(-\frac{3}{2})$. [GBTU 2013]

Lecture 15

Beta function and its properties

Beta function :- Beta function is denoted as $\beta(m,n)$; $m>0,\ n>0$ and defined as

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$

Results :- (1) $\beta(m,n) = \beta(n,m)$.

(2) $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(3) $$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}, \quad m>0,\ n>0$$ [2017-18]

Proof (3) :-

$$\because \Gamma(m) = \int_0^\infty e^{-t}\,t^{m-1}\,dt$$

Putting $t = x^2$, $dt = 2x\,dx$ then

$$\Gamma(m) = 2\int_0^\infty e^{-x^2}x^{2m-1}\,dx \qquad (1)$$

Similarly $$\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}\,dy \qquad (2)$$

then

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty e^{-x^2}x^{2m-1}\,dx\int_0^\infty e^{-y^2}y^{2n-1}\,dy$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}\,dx\,dy$$

Changing into polar coordinates, we have

$$\Gamma(m)\Gamma(n) = 4\int_0^{\pi/2}\int_{r=0}^\infty e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}\,r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_{r=0}^\infty e^{-r^2}r^{2(m+n)-1}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,dr\,d\theta$$

$$= 4\int_0^\infty e^{-r^2}r^{2(m+n)-1}\,dr\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta$$

Put $r^2 = t$, $2r\,dr = dt$

$$= 2\int_0^\infty e^{-t}\,t^{m+n}\frac{dt}{2r^2}\int_0^{\pi/2}2\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta$$

$$= \int_0^\infty e^{-t}\,t^{m+n-1}\,dt\,.\ \beta(n,m)$$

$$= \Gamma(m+n)\,\beta(n,m)$$

$$\beta(n,m) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \beta(m,n)$$

$\because \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$

If we put $x = \sin^2\theta$ then

$$\beta(m,n) = \int_0^{\pi/2} 2\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

Example:- Evaluate $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

Sol:-

$$I = \int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$$

$$= \int_0^{\infty} \frac{x^8}{(1+x)^{24}}\,dx - \int_0^{\infty} \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9,15) - \beta(15,9) = 0$$

$\because \beta(m,n) = \beta(n,m)$

Example:- Prove that $\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$ [2015-16]

Sol:- $I = \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$

Putting $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$, $dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta$

$$I = \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}}\cdot\frac{1}{\sqrt{\tan\theta}}\sec^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sec\theta}\cdot\frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\sec^2\theta\,d\theta = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{1}{\sqrt{\sin 2\theta}}\,d\theta$$

$$= \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2} t\,dt \qquad \begin{matrix} 2\theta = t \\ 2\,d\theta = dt \end{matrix}$$

$$= \frac{1}{2\sqrt{2}}\;\frac{\Gamma\left(\frac{-1/2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{3/2}{2}\right)}$$

$$= \frac{1}{2\sqrt{2}}\;\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

Practice questions:-

(1) Evaluate $\int_0^{\infty} \frac{1}{1+x^4}\,dx$ Ans:- $\frac{\pi\sqrt{2}}{4}$

Lecture 16

Problems based on Beta and Gamma functions

Example:- Show that $\int_0^1 x^5(1-x^3)^{10}dx = \frac{1}{396}$

Sol:- $I = \int_0^1 x^5(1-x^3)^{10}dx$

Putting $x^3 = t \Rightarrow x = t^{1/3}$ and $dx = \frac{1}{3}t^{-\frac{2}{3}}dt$

Now

$$I = \int_0^1 t^{5/3}(1-t)^{10}\frac{1}{3}t^{-\frac{2}{3}}dt$$

$$= \frac{1}{3}\int_0^1 t(1-t)^{10}dt$$

$$= \frac{1}{3}\int_0^1 t^{2-1}(1-t)^{11-1}dt = \frac{1}{3}\beta(2,11)$$

$$= \frac{1}{3}\frac{\Gamma(2)\Gamma(11)}{\Gamma(13)} = \frac{1}{3}\frac{1\times 10!}{12!}$$

$$= \frac{1}{3\times 12\times 11} = \frac{1}{396} \quad \underline{Ans}$$

Example:- For a β function, show that

$$\beta(p,q) = \beta(p+1,q) + \beta(p,q+1)$$

Sol:- R.H.S

$$\beta(p+1,q) + \beta(p,q+1)$$

$$= \frac{\Gamma(p+1)\Gamma(q)}{\Gamma(p+q+1)} + \frac{\Gamma(p)\Gamma(q+1)}{\Gamma(p+q+1)}$$

$$= \frac{p\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)} + \frac{q\Gamma(p)\Gamma(q)}{(p+q)\Gamma(p+q)}$$

$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\left[\frac{p}{p+q} + \frac{q}{p+q}\right]$$

$$= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \beta(p,q)$$

Example:- Show that [2015-16]

$$\frac{\beta(p,q+1)}{q} = \frac{\beta(p+1,q)}{p} = \frac{\beta(p,q)}{p+q}$$

$p>0,\ q>0$

Sol:- Use formula $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ to show the relation.

Example :- Using Beta and Gamma functions, evaluate $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx$ [2017-18]

Sol :-

$$I = \int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx$$

Putting $x^3 = t$, $x = t^{1/3}$ and $dx = \frac{1}{3} t^{-\frac{2}{3}} dt$

$$I = \int_0^1 t^{1/2} (1-t)^{-1/2} \frac{1}{3} t^{-\frac{2}{3}} dt$$

$$= \frac{1}{3} \int_0^1 t^{\left(-\frac{1}{6}\right)} (1-t)^{-1/2} dt$$

$$= \frac{1}{3} \int_0^1 t^{\frac{5}{6}-1} (1-t)^{\frac{1}{2}-1} dt = \frac{1}{3} \beta\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$= \frac{1}{3} \frac{\Gamma(5/6)\, \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}+\frac{1}{2}\right)} = \frac{1}{3} \frac{\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{3}\right)}$$

$$= \frac{\sqrt{\pi}}{3} \frac{\Gamma\left(\frac{5}{6}\right)}{\frac{1}{3}\Gamma\left(\frac{1}{3}\right)} = \frac{\sqrt{\pi}\, \Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}$$ Ans

Example:- Evaluate $I = \int_0^1 \left(\frac{x}{1-x^3}\right)^{1/2} dx$

Ans :- $\frac{\pi}{3}$

Lecture 17
Dirichlet's Integral

Definition :- If l, m, n are all positive, then the triple integral

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\,dy\,dz = a^{l+m+n} \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

where V is the region $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq a$.

if $a = 1$ then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\,dy\,dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}$$

where V is $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$

Example :- Find the mass of a tetrahedron which is formed by the coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the density is given by $\rho = kxyz$. [2016-17]

Sol :- Put $\frac{x}{a} = u$, $\frac{y}{b} = v$, $\frac{z}{c} = w$

then $u \geq 0, v \geq 0, z \geq 0$ $\because x \geq 0, y \geq 0, z \geq 0$

and $u+v+w \leq 1$ be the volume.

also $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$

Volume of the tetrahedron

$$= \iiint_V dx\,dy\,dz$$

$$= \iiint_{V'} abc\,du\,dv\,dw$$

$$= abc \iiint_V u^{1-1} v^{1-1} w^{1-1}\,du\,dv\,dw$$

$$= abc\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(4)}$$

$$= \frac{abc}{6}$$

$$\text{Mass} = \iiint_V k\,xyz\,dx\,dy\,dz$$

$$= \iiint_{V'} k\,(au)(bv)(cw)\,abc\,du\,dv\,dw$$

$$= \iiint_{V'} k\,a^2b^2c^2\,u^{2-1} v^{2-1} w^{2-1}\,du\,dv\,dw$$

$$= \frac{k\,a^2b^2c^2\,\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(7)} = \frac{k\,a^2b^2c^2}{720}$$

Ans

Example :- Evaluate $\iiint_V x^2yz\,dx\,dy\,dz$ throughout the volume bounded by planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Ans :- $\frac{a^3b^2c^2}{2520}$

Example:- Find the mass of the solid $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r \le 1$ and $x\geq 0, y\geq 0, z\geq 0$. The density at any point being $\rho = k\,x^{l-1}y^{m-1}z^{n-1}$.

Sol :- Put $\left(\frac{x}{a}\right)^p = u$, $\left(\frac{y}{b}\right)^q = v$, $\left(\frac{z}{c}\right)^r = w$

$x = a\,u^{1/p}$, $y = b\,v^{1/q}$, $z = c\,w^{1/r}$

$dx = \frac{a}{p}\,u^{\frac{1}{p}-1}$, $dy = \frac{b}{q}\,v^{\frac{1}{q}-1}$, $dz = \frac{c}{r}\,w^{\frac{1}{r}-1}$

Then region becomes

$u\geq 0$, $v\geq 0$, $w\geq 0$ and $u+v+w\le 1$

$$\text{mass} = \iiint_V k\,x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$$

$$\text{mass} = k\iiint_{V'} \left(a u^{1/p}\right)^{l-1}\left(b v^{1/q}\right)^{m-1}\left(c w^{1/r}\right)^{n-1} \times \frac{abc}{pqr}\, u^{\frac{1}{p}-1} v^{\frac{1}{q}-1} w^{\frac{1}{r}-1}\, du\,dv\,dw$$

$$= \frac{a^l b^m c^n}{pqr}\iiint_{V'} u^{\frac{l-1}{p}+\frac{1}{p}-1}\, v^{\frac{m}{q}-\frac{1}{q}+\frac{1}{q}-1}\, w^{\frac{n}{r}-\frac{1}{r}+\frac{1}{r}-1}\, du\,dv\,dw$$

$$= \frac{a^l b^m c^n}{pqr}\iiint_{V'} u^{\frac{l}{p}-1}\, v^{\frac{m}{q}-1}\, w^{\frac{n}{r}-1}\, du\,dv\,dw$$

$$= \frac{a^l b^m c^n}{pqr}\cdot\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$$

Ans

Example:- Find the volume and the mass contained in the solid region in the first octant of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ if the density at any pt is $\rho(x,y,z) = kxyz$ [2019-20]

Ans:- volume $= \frac{\pi abc}{6}$, mass $= \frac{k a^2 b^2 c^2}{48}$

Lecture 18

Liouville's Extension of Dirichlet's integral

Definition:- If the variables x, y, z are all positive such that $h_1 < (x+y+z) < h_2$

then $$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$$

Example:- Show that $$\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8},$$ the integral being extended to all positive values of the variables for which the expression is real. [2015-16]

Sol:- The expression will be real, if

$$1-x^2-y^2-z^2 > 0$$
$$x^2+y^2+z^2 < 1$$

hence the given integral is extended for all positive value of x, y, z such that

$$0 < x^2+y^2+z^2 < 1$$

Now Putting $x^2 = u \Rightarrow x = \sqrt{u}$ and $dx = \frac{1}{2\sqrt{u}} du$

$y^2 = v \Rightarrow y = \sqrt{v}$ and $dy = \frac{1}{2\sqrt{v}} dv$

$z^2 = w \Rightarrow z = \sqrt{w}$ and $dz = \frac{1}{2\sqrt{w}} dw$

then the condition becomes

$$0 < u + v + w < 1$$

Now integral becomes

$$\frac{1}{8} \iiint \frac{u^{-1/2} v^{-1/2} w^{-1/2}}{\sqrt{1-(u+v+w)}} \, du\, dv\, dw$$

$$= \frac{1}{8} \iiint \frac{u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{1}{2}-1}}{\sqrt{1-(u+v+w)}} \, du\, dv\, dw$$

$$= \frac{1}{8} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \int_0^1 \frac{1}{\sqrt{1-t}} t^{\frac{3}{2}-1} dt$$

$$= \frac{1 \times (\sqrt{\pi})^3}{8 \times \frac{1}{2}\Gamma(\frac{1}{2})} \int_0^1 \frac{1}{\sqrt{1-t}} t^{1/2} dt$$

$$= \frac{\pi}{4} \int_0^1 \frac{t^{1/2}}{\sqrt{1-t}} dt$$

Putting $t = \sin^2\theta \Rightarrow dt = 2 \sin\theta \cos\theta \, d\theta$.

$$= \frac{\pi}{4} \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot 2 \sin\theta \cos\theta \, d\theta$$

$$= \frac{\pi}{4} \int_0^{\pi/2} 2 \sin^2\theta \, d\theta = \frac{\pi}{2} \int_0^{\pi/2} \sin^2\theta \cos^0\theta \, d\theta$$

$$= \frac{\pi}{2} \frac{\Gamma(3/2)\Gamma(1/2)}{2\Gamma(2)}$$

$$= \frac{\pi}{4} \cdot \frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{1} = \frac{\pi^2}{8} \text{ Ans}$$

Example :- Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{a^2-x^2-y^2-z^2}}$, the integral being extended to all positive values of the variables for which the expression is real.

Ans :- $\frac{\pi^2 a^2}{8}$

Example :- Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2} = \frac{3}{4} - \log 2$, the integral being taken throughout the volume bounded by the planes

$x=0,\ y=0,\ z=0$ and $x+y+z=1$

Sol:- Acc. to region. $0 \le x+y+z \le 1$

so

$$\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2}$$

$$= \iiint \frac{x^{1-1}\,y^{1-1}\,z^{1-1}}{(x+y+z+1)^2}\,dx\,dy\,dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^2}\,u^{1+1+1-1}\,du$$

$$= \frac{1}{2}\int_0^1 \frac{u^2}{(u+1)^2}\,du$$

Put $u+1 = t \Rightarrow u = t-1$ and $du = dt$

$$= \frac{1}{2}\int_1^2 \frac{(t-1)^2}{t^2}\,dt$$

$$= \frac{1}{2}\int_1^2 \frac{t^2+1-2t}{t^2}\,dt$$

$$= \frac{1}{2}\left[\int_1^2 dt + \int_1^2 t^{-2}\,dt - 2\int_1^2 t^{-1}\,dt\right]$$

$$= \frac{1}{2}\left[1 + \left(-t^{-1}\right)_1^2 - 2\left(\log t\right)_1^2\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + 1 - 2(\log 2)\right] = \frac{3}{4} - \log 2$$

10 Years AKTU University Examination Questions			Unit-4
S. No	Questions	Session	Lecture No
1.	Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}} dxdy$	2011-12 (short)	23-32
2.	Prove that $\iint \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} ds = \frac{4\pi}{\sqrt{abc}}$, where S is the ellipsoid $ax^2+by^2+cz^2=1$	2011-12 (short)	23-32
3.	Evaluate $\iint (x-y)^4 \exp(x+y)\, dxdy$ where R is the square in the xy-plane with vertices at (1,0), (2,1), (1,2) and (0,1)	2012-13 (short)	23-32
4.	Evaluate $\int_0^\infty\int_0^x x.\exp\left(\frac{-x^2}{y}\right) dxdy$	2012-13 (long)	23-32
5.	Evaluate $\int_0^a\int_0^x xydydx$	2013-14 (short)	23-32
6.	Evaluate $\int_0^1\int_0^1 \frac{dxdy}{\sqrt{1-x^2}\sqrt{1-y^2}}$	2015-16 (short)	23-32
7.	Evaluate $\int_0^1\int_0^{x^2} xe^y dxdy$	2017-18 (short)	23-32
8.	Evaluate $\int_0^1\int_0^{x^2} e^{\frac{y}{x}}dxdy$	2018-19 (short)	23-32
9.	Evaluate $\int_0^2\int_0^1 (x^2+3y^2)dydx$	2019-20 (short)	23-32
10.	Compute the area bounded by lemniscate$r^2=a^2\cos 2\theta$	2013-14 (long)	23-32
11.	Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dxdy$by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$	2018-19 (short)	23-32
12.	Changing the order of integration in the double integral: $I=\int_0^8\int_{x/4}^2 f(x,y)dxdy$ leads to $I=\int_r^s\int_p^q f(x,y)dxdy$ say, what is p?	2012-13 (short)	23-32
13.	Change the order of integration and evaluate $\int_0^1\int_{x^2}^{2-x} xydydx$	2014-15 (short) 2015-16 (short) 2016-17 (short) 2017-18 (short) 2019-20 (short)	23-32
14.	Changing the order of integration in the double integral: $I=\int_0^8\int_{x/4}^2 f(xy)dydx$ leads to	2016-17 (long)	23-32

	$I=\int_r^s\int_p^q f(xy)dxdy$ say, what is q?		
15.	Change the order of integration and evaluate $\int_0^2\int_{\frac{x^2}{4}}^{3-x} xydydx$	2018-19 (long)	23-32
16.	Evaluate the following integral by changing the order of integration $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dydx$	2020-21 (long)	23-32
17.	Find the value of the integral $\iint xydxdy$ where R is the region bounded by the x-axis, the line $y=2x$ and the parabola $x^2=4ay$	2011-12 (short)	23-32
18.	Determine the area bounded by the curves $xy=2, 4y=x^2, y=4$	2014-15 (short)	23-32
19.	Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$and the coordinate planes.	2011-12 (long)	23-32
20.	Find the volume of the solid which is bounded by the surfaces $2z=x^2+y^2$ and $z=x$	2011-12 (long)	23-32
21.	Find the volume contained in the solid region in the first octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	2013-14 (short)	23-32
22.	Find the volume and the mass contained in the solid region in the first octant of the ellipsoid$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ if the density at any point $\rho(x,y,z)=kxyz$	2014-15 (long)	23-32
23.	Prove that $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}=\frac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real.	2015-16 (short)	23-32
24.	Evaluate $\iiint (x+y+z)dxdydz$ where $R: 0\le x\le 1; 1\le y\le 2; 2\le z\le 3$	2015-16 (long) 2017-18 (long)	23-32
25.	If the volume of an object expressed in the spherical coordinates as following: $$V=\int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2\sin\theta drd\theta d\phi$$ Evaluate the value of V.	2016-17 (short)	23-32
26.	Evaluate the triple integral$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyzdxdydz$	2016-17 (short)	23-32
27.	Evaluate $\iiint x^2yz\, dxdydz$ throughout the volume bonded by planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$	2016-17 (long)	23-32
28.	Calculate the volume of the solid bounded by the surface $x=0, y=0, z=0$ and $x+y+z=1$	2018-19 (short) 2020-21 (short)	23-32
29.	Find the volume of the largest rectangular parallelopiped that	2019-20 (short)	23-32

	can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$		
30.	Find the volume of the region bounded by the surface $y=x^2, x=y^2$ and the plane $z=0, z=3$.	2019-20 (long)	23-32
31.	Evaluateby changing the variables $\iint (x+y)^2 dxdy$ where R is the region bounded by the parallelogram $x+y=0, x+y=2, 3x-2y=0, 3x-2y=3$	2013-14 (long) 2020-21 (long)	23-32
32.	Evaluate $\iint (x+y)^2 dxdy$ where R is the region bounded by the parallelogram in the xy-plane with vertices (1,0), (3,1), (2,2), (0,1) using the transformation $u=x+y, v=x-2y$.	2019-20 (long)	23-32

S. No	Questions	Session
1	(a) Evaluate $\frac{\Gamma(8/3)}{\Gamma(2/3)}$	[2015-16]
	(b) Evaluate $\Gamma\left(\frac{-5}{2}\right)$	[2013-14]
2	(a) Find the value of integral $\int_0^\infty e^{-ax}x^{n-1}dx$	[2015-16]
	(b)Evaluate $\Gamma(3/4)\Gamma(1/4)$	[2012-13]
3	Prove that $\sqrt{\pi}\,\Gamma(2n)=2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)$.	[2011-12]
4	(a) For the Gamma function, show that $\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)}=(2)^{1/3}\sqrt{\pi}$.	[2016-17]
	(b) Show that $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta=\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta=\frac{\pi}{\sqrt{2}}$	
5	(a) Prove that $\int_0^1 \frac{1}{\sqrt{1+x^4}}dx=\frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$	[2015-16]
	(b) Evaluate $\int_0^\infty \frac{1}{1+x^4}dx$.	[2012-13]
6	Prove that $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m>0$, $n>0$, where Γ is Gamma function.	[2017-18]

7	Use Beta function to evaluate: $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$.	[2011-12]
8	Show that $\int_0^1 x^5(1-x^3)^{10}dx=\frac{1}{396}$	
9	(a) For a β function, show that $\beta(p,q)=\beta(p+1,q)+\beta(p,q+1)$	
	(b)Show that $\frac{\beta(p,q+1)}{q}=\frac{\beta(p+1,q)}{p}=\frac{\beta(p,q)}{p+q}$ where $p>0, q>0$	[2015-16]
10	Using Beta and Gamma functions, evaluate $\int_0^1\left(\frac{x^3}{1-x^3}\right)^{1/2}dx$	[2017-18]
11	Evaluate $I=\int_0^1\left(\frac{x}{1-x^3}\right)^{1/2}dx$	[2013-14]
12	Apply Dirichlet integral to find the volume of an octant of the sphere $x^2+y^2+z^2=25$	[2018-19
13	Find the volume and mass of a tetrahedron which is formed by the co-ordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ the density is given by $\rho=kxyz$	[2017-18]
14	Evaluate $\iiint x^2yz\,dxdydz$ through out the volume bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$	[2016-17]
15	Find the volume and the mass contained in the solid region in the first octant of the ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ if the density at any point $\rho(x,y,z)=kxyz$	[2019-20] [2014-15]
16	Find the mass of the solid $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r=1$, where x, y, z are all positive and the density at any point being $\rho=kx^{l-1}y^{m-1}z^{n-1}$.	[2013-14] [2012-13]
17	Show that $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}=\frac{\pi^2}{8}$, the integral being extended to all positive values of the variables	[2015-16] [2012-13]

	for which the expression is real.	
18	Evaluate $\iiint \frac{dxdydz}{\sqrt{a^2-x^2-y^2-z^2}}$, the integral being extended to all positive values of the variables for which the expression is real.	[2012-13]
19	Show that $\iiint \frac{dxdydz}{(x+y+z+1)^2} = \frac{3}{4} - \log 2$ the integral being taken throughout the volume bounded by the planes $x = 0, y = 0, z = 0 \text{ and } x + y + z = 1$	